魏征高等数学公式整理

一、指数律及对数律:

$e^{x+y}=e^x e^y$

$\ln xy=\ln x+\ln y\ ,\ x,y>0$

$e^{x-y}=\dfrac{e^x}{e^y}$

$\ln\dfrac{x}{y}=\ln x-\ln y\ ,\ x,y>0$

$e^{xy}=(e^x)^y$

$\ln x^y=y\ln x\ ,\ x>0$

$e^{\frac{x}{n}}=\sqrt[n]{e^x}$

$\log_a x=\dfrac{\log_e x}{\log_e a}=\dfrac{\ln x}{\ln a}\ ,\ x>0$

$e^{\ln x}=x$

$\log x=\dfrac{\ln x}{\ln 10}\ ,\ x>0$

$e^{\ln f(x)}=f(x)$ (换底公式)

$\ln e^x=x$

$$\ln\frac{x_1^a x_2^b x_3^c}{x_4^d x_3^e}=a\ln|x_1|+b\ln|x_2|+c\ln|x_3|-d\ln|x_4|-e\ln|x_5|$$

二、因式分解公式:

$x^2-y^2=(x-y)(x+y)$

$x^3-y^3=(x-y)(x^2+xy+y^2)$

$x^3+y^3=(x+y)(x^2-xy+y^2)$

$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\ldots+xy^{n-2}+y^{n-1})$

三、三角函数:

1. $\sin^2 x+\cos^2 x=1$, $1+\tan^2 x=\sec^2 x$, $1+\cot^2 x=\csc^2 x$

2. $\sin 2x=2\sin x\cos x$,

$\cos 2x=\cos^2 x-\sin^2 x=1-2\sin^2 x=2\cos^2-1$

3. $\sin^2\frac{x}{2}=\frac{1-\cos x}{2}$, $\cos^2\frac{x}{2}=\frac{1+\cos x}{2}$ （半角公式）

4. $\sin(x\pm y)=\sin x\cos y\pm\sin y\cos x$

5. $\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y$

6. $2\sin x\cos y=\sin(x+y)+\sin(x-y)$

7. $2\cos x\cos y=\cos(x+y)+\cos(x-y)$

8. $2\sin x\sin y=\cos(x-y)-\cos(x+y)$

9. $\sqrt{1\pm\sin x}=\left|\sin\frac{x}{2}\pm\cos\frac{x}{2}\right|$

$\sqrt{1+\cos x}=\sqrt{2}\left|\cos\frac{x}{2}\right|$, $\sqrt{1-\cos x}=\sqrt{2}\left|\sin\frac{x}{2}\right|$

10. $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$, $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$

11. $\frac{d}{dx}\sin x=\cos x$ $\quad$ $\frac{d}{dx}\csc x=-\csc x\cot x$

$\frac{d}{dx}\cos x=-\sin x$ $\quad$ $\frac{d}{dx}\sec x=\sec x\tan x$

$\frac{d}{dx}\tan x=\sec^2 x$ $\quad$ $\frac{d}{dx}\cot x=-\csc^2 x$

12. $\sin(\frac{\pi}{2}-x)=\cos x$, $\sin(\pi-x)=\sin x$, $\sin(\frac{3\pi}{2}-x)=-\cos x$

$\sin(\frac{\pi}{2}+x)=\cos x$, $\sin(\pi+x)=-\sin x$, $\sin(\frac{3\pi}{2}+x)=-\cos x$

$\cos(\frac{\pi}{2}-x)=\sin x$, $\cos(\pi-x)=-\cos x$, $\cos(\frac{3\pi}{2}-x)=-\sin x$

$\cos(\frac{\pi}{2}+x)=-\sin x$, $\cos(\pi+x)=-\cos x$, $\cos(\frac{3\pi}{2}+x)=\sin x$

13.

角度	$\sin x$	$\cos x$	$\tan x=\dfrac{\sin x}{\cos x}$	$\csc x=\dfrac{1}{\sin x}$	$\sec x=\dfrac{1}{\cos x}$	$\cot x=\dfrac{\cos x}{\sin x}$
0	0	1	0	∞	1	∞
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{2}$	1	0	∞	1	∞	0

四、双曲线函数：

1. $\sinh x=\dfrac{e^x-e^{-x}}{2}$ ， $\cosh x=\dfrac{e^x+e^{-x}}{2}$

$\operatorname{csc}\mathrm{h}x=\dfrac{1}{\sinh x}$ ， $\mathrm{sech}x=\dfrac{1}{\cosh x}$

$\tanh x=\dfrac{\sinh x}{\cosh x}$ ， $\coth x=\dfrac{\cosh x}{\sinh x}$

2. $\coth^2 x-\mathrm{csch}^2 x=1$ ， $\tanh^2 x+\sec\mathrm{h}^2 x=1$

3. $\cosh^2 x-\sinh^2 x=1$ ， $\coth^2 x-\mathrm{csch}^2 x=1$

4. $\dfrac{d}{dx}\sinh x=\cosh x$ ， $\dfrac{d}{dx}\mathrm{csch}x=-\mathrm{csch}x\coth x$

$\dfrac{d}{dx}\cosh x=\sinh x$ ， $\dfrac{d}{dx}\mathrm{sech}x=-\mathrm{sech}x\tanh x$

$\dfrac{d}{dx}\tanh x=\sec\mathrm{h}^2 x$ ， $\dfrac{d}{dx}\coth x=-\mathrm{csch}^2 x$

五、反函数：

1. $\sin(\sin^{-1}x)=x$， $\sin^{-1}(\sin x)=x$， $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$

2. $\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$

$\frac{d}{dx}\cos^{-1}x=\frac{-1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{1+x^2}}$

3. $f^{-1}\cdot f(x)=x$, $f\cdot f^{-1}(x)=x$

4. 若 $f^{-1}(x)=g(x)\Rightarrow g(f(x))=x$ 或 $f(g(x))=x$

六、根号与绝对值：

1. $\sqrt{x^2}=|x|$, $\sqrt{g^2(x)}=|g(x)|$

2. $\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$, $\frac{d}{dx}\sqrt{g(x)}=\frac{1}{2\sqrt{g(x)}}\cdot g'(x)$

3. $\frac{d}{dx}|x|=\frac{|x|}{x}=\frac{x}{|x|}$, $\frac{d}{dx}|g(x)|=\frac{d}{dx}\sqrt{g^2(x)}=\frac{2g(x)}{2\sqrt{g^2(x)}}\cdot g'(x)=\frac{g(x)}{|g(x)|}g'(x)$

七、微分公式：

1. $\frac{d}{dx}a^x=a^x\ln a$

2. $\frac{d}{dx}x^x=x^x(\ln x+1)$

3. $\frac{d}{dx}\sin g(x)=\cos g(x)\cdot g'(x)$, $\frac{d}{dx}\csc g(x)=-\csc g(x)\cot g(x)\cdot g'(x)$

$\frac{d}{dx}\cos g(x)=-\sin g(x)\cdot g'(x)$, $\frac{d}{dx}\sec g(x)=-\sec g(x)\tan g(x)\cdot g'(x)$

$\frac{d}{dx}\tan g(x)=\sec^2 g(x)\cdot g'(x)$, $\frac{d}{dx}\cot g(x)=-\csc^2 g(x)\cdot g'(x)$

4. $\frac{d}{dx}\sinh x=\cosh x$, $\frac{d}{dx}\text{csch}x=-\text{csch}x\coth x$

$\frac{d}{dx}\cosh x=\sinh x$, $\frac{d}{dx}\text{sech}x=-\text{sech}x\tanh x$

$\frac{d}{dx}\tanh x=\text{sech}^2x$, $\frac{d}{dx}\coth x=-\text{csch}^2x$

5. $\frac{d}{dx}\sin^{-1}g(x)=\frac{1}{\sqrt{1-g(x)^2}}\cdot g'(x)$, $\frac{d}{dx}\cos^{-1}g(x)=\frac{-1}{\sqrt{1-g(x)^2}}\cdot g'(x)$

$$\frac{d}{dx}\tan^{-1}g(x)=\frac{1}{1+g(x)^2}\cdot g'(x)\ ,\quad \frac{d}{dx}\cot^{-1}g(x)=\frac{-1}{1+g(x)^2}\cdot g'(x)$$

6. $$\frac{d}{dx}g(x)^n=ng(x)^{n-1}\cdot g'(x)$$

$$\frac{d}{dx}e^{g(x)}=e^{g(x)}\cdot g'(x)$$

$$\frac{d}{dx}\ln g(x)=\frac{1}{g(x)}\cdot g'(x)$$

八、几何公式：

1. $A(a_1,a_2)$, $B(b_1,b_2)$ $\Rightarrow$ $\overline{AB}=\sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$

2. 圆柱体表面积 $=2\pi r^2+2\pi rh$ r : 半径 ， h : 高

3. 球体表面积 $=4\pi r^2$

4. 圆锥体的表面积 $=\pi r\sqrt{r^2+h^2}+\pi r^2$

5. 圆柱体积 $=\pi r^2 h$

6. 球体体积 $=\frac{4}{3}\pi r^3$

7. 圆锥体体积 $=\frac{1}{3}\pi r^2 h$

九、积分公式及重要结果：

1. $\int \sec^2 xdx=\tan x+c$, $\int \csc^2 xdx=-\cot x+c$

2. $\int \sec x\tan xdx=\sec x+c$, $\int \csc x\cot xdx=-\csc x+c$

3. $\int x^a dx=\frac{1}{a+1}x^{a+1}+c$

4. $\int a^x dx=\frac{1}{\ln a}a^x+c$

5. $\int xe^{ax^2}dx=\frac{1}{2a}e^{ax^2}+c$

6. $\int \frac{g(x)}{g'(x)}dx = \ln|g(x)| + c$

7. $\int \tan x dx = -\ln|\cos x| + c$, $\int \cot x dx = -\ln|\sin x| + c$

8. $\int \cos x \sin^n x dx = \frac{1}{n+1}\sin^{n+1} x + c$, $\int \sin x \cos^n x dx = -\frac{1}{n+1}\cos^{n+1} x + c$

9. $\int \sec^2 x \tan^n x dx = \frac{1}{n+1}\tan^{n+1} x + c$, $\int \tan x \sec^n x dx = \frac{1}{n}\sec^n x + c$

10. $\int xe^x dx = (x-1)e^x + c$, $\int xe^{-x} dx = -(x+1)e^{-x} + c$

11. $\int \ln x dx = x\ln x - x + c$

12. $\int \sec x dx = \ln|\sec x + \tan x| + c$

$\int \csc x dx = -\ln|\csc x + \cot x| + c = \ln|\csc x - \cot x| + c$

13. $\int \frac{1}{1+e^x}dx = \int \frac{e^{-x}}{e^{-x}+1}dx = -\ln|1+e^{-x}| + c$

$\int \frac{1}{1+e^{-x}}dx = \int \frac{e^x}{e^x+1}dx = \ln|1+e^x| + c$

14. $\int e^{ax}\cos bx dx = \frac{1}{a^2+b^2}(a\cos bx + b\sin bx)e^{ax} + c$

$\int e^{ax}\sin bx dx = \frac{1}{a^2+b^2}(a\sin bx - b\cos bx)e^{ax} + c$

15. $\int \sec^3 x dx = \frac{1}{2}\tan x \sec x + \frac{1}{2}\ln|\sec x + \tan x| + c$

16. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$, $\int \frac{1}{(x+b)^2+a^2}dx = \frac{1}{a}\tan^{-1}(\frac{x+b}{a}) + c$

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$

17. 三角代换（II）：令 $u = \tan\frac{x}{2}$ 則 $\begin{cases} \sin x = \frac{2u}{1+u^2} \\ \cos x = \frac{1-u^2}{1+u^2} \\ dx = \frac{2}{1+u^2}du \end{cases}$

十、极限：

1. $\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$ $\lim\limits_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$

2. $\lim\limits_{n\to\infty}\sqrt[n]{n} = 1$

3. $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ $\lim\limits_{x\to\pm\infty}\frac{\sin x}{x} = 0$

$\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$ $\lim\limits_{x\to\pm\infty} x\sin\frac{1}{x} = 1$

4. $\lim\limits_{x\to 0^+} x^a \ln x = 0$ ， $a > 0$ $\lim\limits_{x\to 0^+} x^x = 1$

5. $\lim\limits_{x\to\infty} = (1+\frac{1}{x})^x = e$

十一、定积分公式：

1. 微积分基本定理

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f(g_2(x))g'_2(x) - f(g_1(x))g_1'(x)$$

2. $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$ 即 $\sin x \rightleftarrows \cos x$ 函數值不變

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^n x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^n x}dx = \frac{\pi}{4}$$

3. Gamma 函数 $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}dx = n\Gamma(n) = ... = n!$

十二、周期：

1. $\sin x$ ， $\cos x$ $p = 2\pi$

2. $\tan x$ ， $\cot x$ $p = \pi$

3. $\sin ax$ ， $\cos ax$ $\qquad p=\dfrac{2\pi}{a}$

4. $|\sin ax|$, $|\cos ax|$ $\qquad p=\dfrac{\pi}{a}$

十三、三角函数的积分公式：

1. $\int_0^{\frac{\pi}{2}} f(\sin x,\cos x)dx=\int_0^{\frac{\pi}{2}} f(\cos x,\sin x)dx$

2. $\int_0^{\pi} f(\sin x)dx=2\int_0^{\frac{\pi}{2}} f(\sin x)dx$

3. $\int_0^{\pi} xf(\sin x)dx=\dfrac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

4. Wallis 公式：

$$\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\cos^n xdx=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & n\text{ 是偶數}\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3}\cdot 1, & n\text{ 是奇數}\end{cases}$$

十四、坐标转换：

1. 极坐标 $(x,y)\to(r,\theta)$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\qquad \begin{cases}r^2=x^2+y^2\\ \theta=\tan^{-1}\dfrac{y}{x}\end{cases}\qquad 0\le\theta\le 2\pi\ ,\quad J=r$$

2. 圆柱坐标 $(x,y,z)\to(\rho,\theta,z)$

$$\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\\ z=z\end{cases}\qquad \begin{cases}x^2+y^2=\rho^2\\ z=z\end{cases}\qquad 0\le\theta\le 2\pi\ ,\quad J=\rho$$

3. 球坐标 $(x,y,z)\to(r,\phi,\theta)$

$$\begin{cases}x=r\sin\phi\cos\theta\\ y=r\sin\phi\sin\theta\\ z=r\cos\phi\end{cases}\qquad x^2+y^2+z^2=r^2$$

$0\le\phi\le\pi$ ， $0\le\theta\le 2\pi$ ， $J=r^2\sin\phi$

十五、级数：

1. 等比级数 $a_1+a_2+\ldots a_n=a_1+a_1r+\ldots+a_1r^{n-1}=\dfrac{a_1(1-r^n)}{1-r}$ ，$a_n=a_1r^{n-1}$

2. $\dfrac{1}{1-x}=1+x+x^2+\ldots..+x^k+\ldots..=\sum\limits_{k=0}^{\infty}x^k$ ， $|x|<1$

3. $\tan^{-1}x=x-\dfrac{1}{3}x^3+\dfrac{1}{5}x^5\ldots..+\dfrac{(-1)^k}{2k+1}x^{2k+1}+\ldots=\sum\limits_{k=0}^{\infty}\dfrac{(-1)^k}{2k+1}x^{2k+1}$

4. $e^x=1+x+\dfrac{1}{2!}x^2+\ldots..+\dfrac{1}{k!}x^k+\ldots=\sum\limits_{k=0}^{\infty}\dfrac{1}{k!}x^k$

5. $\sin x=x-\dfrac{1}{3!}x^3+\dfrac{1}{5!}x^5\ldots...+\dfrac{(-1)^k}{(2k+1)!}x^{2k+1}+\ldots=\sum\limits_{k=0}^{\infty}\dfrac{(-1)^k}{(2k+1)!}x^{2k+1}$

6. $\cos x=1-\dfrac{1}{2!}x^2+\dfrac{1}{4!}x^4+\ldots..+\dfrac{(-1)^k}{(2k)!}x^{2k}+\ldots=\sum\limits_{k=0}^{\infty}\dfrac{(-1)^k}{(2k)!}x^{2k}$

7. $(1+x)^{\alpha}=1+\alpha x+\dfrac{\alpha(\alpha-1)}{2!}x^2+\ldots+\dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}x^k+\ldots$

$$=1+\sum_{k=1}^{\infty}\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}x^k$$

8. $f^n(c)=n!\times a_n$ ， a_n 为 $(x-c)^n$ 前面的系数

9. $\dfrac{d^{2n+1}}{dx^{2n+1}}\sin^{-1}x\Big|_{x=0}=[1\cdot 3\cdots(2n-1)]^2$ ， $n=0,1,2,3,\ldots..$

十六、重积分：

1. $\int_{-\infty}^{\infty}e^{-ax^2}dx=2\int_0^{\infty}e^{-ax^2}dx=\sqrt{\dfrac{\pi}{a}}$

2. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}f(x,y)dydx=\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}f(x,y)dxdy=\iint\limits_{x^2+y^2\le a^2}f(x,y)dA$

3. $\int_{-a}^{a}\int_{\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}}\int_{\sqrt{a^2-y^2-z^2}}^{\sqrt{a^2-y^2-z^2}}f(x,y,z)dxdydz=\iiint\limits_{x^2+y^2+z^2\le a^2}f(x,y)dV$

十七、向量：

1. 单位向量 $\vec{u}=\dfrac{\overrightarrow{RS}}{|\overrightarrow{RS}|}$

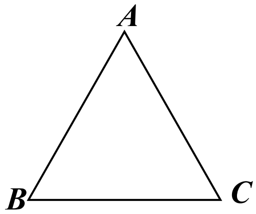
2. $\vec{A} = [a_1, a_2, a_3]$ ， $\vec{B} = [b_1, b_2, b_3]$ ， θ 爲 $\vec{A} \cdot \vec{B}$ 夾角

向积 $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$

外积 $\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

3. $p\,(x_0, y_0, z_0)$ 至 $ax + by + cz = d$ 之距离

$$d = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$



4. 面积 $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

十八、近似值：

1. $f(x + \Delta x) \fallingdotseq f(x) + f'(x)\Delta x$

2. $f(x + \Delta x，y + \Delta y) \fallingdotseq f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y$

3. $f(x + \Delta x) \fallingdotseq f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + ...$ (Taylor 展开)

4. Simpson’s rule

$$\int_a^b f(x)dx$$

$$\fallingdotseq \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 4f(x_{n-1}) + f(x_n)]$$

$$x_0 = a，x_1 = a + \frac{b-a}{n}，x_2 = a + \frac{2(b-a)}{n}，......，x_n = b,\quad \Delta x = \frac{b-a}{n}$$

5. 牛顿公式：(求近似根) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$